Pergamon

Int. J. Heat Mass Transfer. Vol. 39, No. 13, pp. 2821–2829, 1996

0017–9310/96 $15.00+0.00

0017–9310(95)00355–X

Weakly nonlinear stability analysis of condensate film flow down a vertical cylinder

CHEN-I HUNG, CHA'O-KUANG CHEN† and JUNG-SHUN TSAI
Department of Mechanical Engineering, National Cheng-Kung University, Tainan, Taiwan, 301, Republic of China

(*Received* 4 *June* 1995)

Abstract—Weakly nonlinear stability theory is used to study the stability characteristics of condensate film flowing down the outer surface of a vertical cylinder. The surface tension and the mass transfer due to phase change are taken into account at the liquid–vapor interface. A method of perturbation is applied for the solution and the results show that supercritical stability in the linearly unstable region and subcritical instability in the linearly stable region exist. The lateral curvature of the cylinder has a destabilizing effect on the flow stability. The curvature of the cylinder will intensify the instability of the film flow in comparison with that of the planar flow. A possible application of the present results to some aspects of the qualitative design of a coating process is given. Copyright © 1996 Elsevier Science Ltd.

INTRODUCTION

The instability problem of fluid film flow down a vertical or inclined plate is commonly found in many industrial applications, such as for coating, laser cutting process and casting technology. It would be highly desirable to know the flow configuration and its time dependence in order to develop suitable conditions under which the homogeneous film growth could be obtained.

The theory of laminar film condensation flow due to gravity was first developed by Nusselt [1], but the stability of condensate film flow had never been investigated until the 1970s. Bankoff [2], Marschall and Lee [3], and Lin [4] successively investigated the stability analysis of a condensate film flowing down a vertical or an inclined plate. They concluded that the critical Reynolds number is small for all practical condensation problems, and therefore, the liquid film can be considered to be unstable. Also, they pointed out that condensation will stabilize the film flow, while evaporation will destabilize the flow. Ünsal and Thomas [5] presented a linear stability analysis of condensate film flow. They considered the effect of mass transfer at the liquid–vapor interface. Burelbach *et al.* [6] considered a thin viscous liquid layer which is bounded above by its vapor and below by a uniformly heated (or cooled) rigid plane surface, they derived a one-sided model to decouple the dynamics of the vapor from that of liquid, and discussed the instability due to the effects of mass loss or gain, vapor recoil, thermocapillarity, long-range molecular forces, surface tension, and viscous forces. In fact, the boundary conditions used by Ünsal and Thomas [5] are the special cases of Burelbach *et al.* [6]. Joo *et al.* [7] extended the work of Burelbach *et al.* [6] to include the effect of gravity and hydrostatic pressure. Essentially, the linear stability analysis can only be applied to study the cases of infinitesimal disturbances. When disturbance grows to be of a finite value, linear stability theory becomes inappropriate. The nonlinear modification of linear waves was studied by Benney [8], but the effect of surface tension was not included, so the solution had no tendency toward a finite-amplitude equilibrium state. While considering the effect of surface tension, supercritical stability was found to be possible [9, 10]. Ünsal and Thomas [11] investigated the nonlinear stability of vertical condensate film flow by using perturbation methods. Hwang and Weng [12] showed that both supercritical stability and subcritical instability are possible for condensate film flow. Tsai *et al.* [13] studied the nonlinear stability of electrically conducting liquid film under the action of magnetic field. An extensive review article on the evolution of falling film wave problems can be found in Chang [14].

Hydrodynamic stability problems regarding film flowing down a vertical cylinder surface has been studied by some researchers. Lin and Liu [15] compared their theoretical results with known experiments of falling films on cylinder and of creeping annular threads in viscous liquid. Krantz and Zollars [16] presented an asymptotic solution to point out the important effect of curvature on the stability of the film flow, and showed that the curvature of the cylinder will intensify the instability of film flow in comparison with the planar flow. Lin and Weng [17] dealt with the linear stability of condensate film flow down a vertical cylinder and considered the effect of phase changes, but some errors existed in the coefficients of their derived generalized kinematic evolution equa-

† Author to whom correspondence should be addressed.

NOMENCLATURE

a	$= r_0/h_0$, dimensionless radius distance
C_p	specific heat of liquid
d	$d_r + id_i$, complex wave celerity
g	gravitational acceleration
h	film thickness
h_o	local base flow film thickness
h_{fg}	latent heat
K	liquid thermal conductivity
Nd	$= (1-\beta)\xi^2/\beta Pr^2$, dimensionless
W	$= (S^3/2^4\rho^3\nu^4 g)^{1/3}$, dimensionless surface tension
p	liquid pressure
p_g	vapor pressure
Pe	$= Pr\ Re$, local Peclet number
Pr	$= \rho\nu C_p/K$, Prandtl number
r_0	dimension radius
Re	$= u_o h_o/\nu$, Reynolds number
t	time
S	surface tension of liquid
T	liquid temperature
T_s	vapor saturation temperature
T_w	wall temperature
u_o	$= gh_0^2/4\nu\Gamma$, reference velocity
u, w	velocities along z- and r-directions, respectively
z, r	coordinates along and transverse to the cylinder surface.

Greek symbols

α	wave number
β	$= \rho_g/\rho$, density ratio
ε	infinitesimal parameter
ξ	$= C_p(T_s - T_w)/h_{fg}$, Jakob number
η	dimensionless disturbance film thickness
θ	dimensionless temperature
ν	fluid kinematic viscosity
μ	fluid dynamic viscosity
ρ	liquid density
ρ_g	vapor density
φ	liquid stream function
λ	disturbance wavelength.

Superscripts

*	dimensionless quantities.

tion for film thickness. Rosenau and Oron [18] derived an amplitude equation which describes the evolution of a disturbed film interface flowing down an infinite vertical cylindrical column. They pointed out numerically that both supercritical stability and subcritical instability are possible for film flow, they showed also that the evolving waves may break in a finite time for some linearly unstable equilibrium. Davalos-Orozco and Ruiz-Chavarria [19] investigated the linear stability of a fluid layer flowing down the inside and outside of a rotating vertical cylinder; they pointed out that the centrifugal force can stabilize the film flow so as to counteract the destabilizing effect of surface tension. In the absence of rotation, the stability still can be found for some critical wave number.

In this present study, the corrected model for the linear stability analysis of film flow proposed by Lin and Weng [17] is extended to investigate the weakly nonlinear instability of condensate film flowing down the outer surface of a vertical cylinder. The method of multiple scales is applied to solve the nonlinear generalized kinematic equation order by order, and a secular equation of the Ginzburg–Landau type is obtained. Through the nonlinear analysis, we could realize theoretically that the equilibrium finite amplitude could be controlled by the adjustment of relevant parameters, such as the radius of cylinder.

ANALYSIS

Consider the condensate film flow of an incompressible viscous film on the outer surface of an infinite vertical cylinder, as shown in Fig. 1. Under the assumption of constant physical properties and axisymmetric configuration, the basic equations governing the flow can be written as follows:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \quad (1)$$

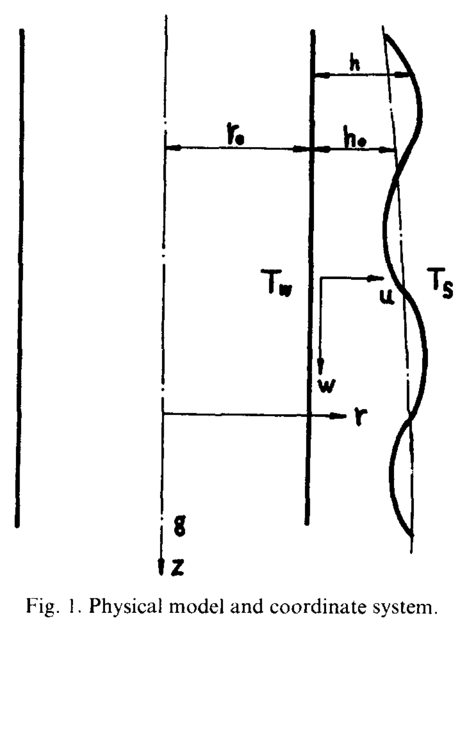

Fig. 1. Physical model and coordinate system.

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial r}+v\left(\nabla^2-\frac{1}{r^2}\right)u, \quad (2)$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p}{\partial z}+\frac{1}{v}\nabla^2 w+g, \quad (3)$$

$$\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial r}+w\frac{\partial T}{\partial z}=\frac{K}{\rho C_p}\nabla^2 T \quad (4)$$

where

$$\nabla^2=\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{\partial^2}{\partial z^2},$$

u and w are the velocity components along the r- and z-directions, respectively, p the liquid pressure, and T the temperature. The last term of equation (2) denotes the body force due to gravity.

The boundary conditions at the outer wall of the cylinder are the no-slip condition for velocities and a constant wall temperature, i.e.

$$r=r_o,$$

$$u=w=0, \quad T=T_w. \quad (5)$$

While at the liquid–vapor interface, the balance of the normal and tangential stresses, the balance of the interfacial energy, and the equality of liquid and vapor temperatures are given as:

At free surface ($r=r_o+h$)

$$p-p_g+K^2\left(\frac{\partial T}{\partial r}-\frac{\partial h}{\partial z}\frac{\partial T}{\partial z}\right)^2 \times h_{fg}^{-2}\rho^{-1}\beta^{-1}(\beta-1)\left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]^{-1} -2\rho v\left[\frac{\partial u}{\partial r}-\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)\frac{\partial h}{\partial z} +\frac{\partial w}{\partial z}\left(\frac{\partial h}{\partial z}\right)^2\right]\left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]^{-1} +S\frac{\partial^2 h}{\partial z^2}\left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]^{3/2}-S\frac{1}{r}\left[1+\left(\frac{\partial h}{\partial z}\right)^2\right]^{-1/2}=0, \quad (6)$$

$$\left(\frac{\partial w}{\partial r}+\frac{\partial u}{\partial z}\right)+2\left(\frac{\partial w}{\partial z}-\frac{\partial u}{\partial r}\right)\frac{\partial h}{\partial z}\left[\left(\frac{\partial h}{\partial z}\right)^2-1\right]^{-1}=0, \quad (7)$$

$$K\left(\frac{\partial T}{\partial r}-\frac{\partial T}{\partial z}\frac{\partial h}{\partial z}\right)+\rho h_{fg}\left(u-w\frac{\partial h}{\partial z}-\frac{\partial h}{\partial t}\right)=0, \quad (8)$$

$$T=T_s. \quad (9)$$

Introduce the stream function φ such that

$$u=\frac{1}{r}\frac{\partial\varphi}{\partial z} \quad w=-\frac{1}{r}\frac{\partial\varphi}{\partial r} \quad (10)$$

and define the following dimensionless quantities:

$$p^*=\frac{p-p_g}{\rho u_o^2} \quad \varphi^*=\frac{\varphi}{u_o h_o^2} \quad h^*=\frac{h}{h_o} \quad \theta=\frac{T-T_w}{T_s-T_w}$$

$$r^*=\frac{r}{h_o} \quad a=\frac{r_o}{h_o} \quad z^*=\frac{\alpha z}{h_o} \quad t^*=\frac{\alpha u_o t}{h_o}$$

$$Re=\frac{u_o h_o}{v} \quad Pr=\frac{\rho v C_p}{K} \quad Pe=PrRe,$$

$$W=\left(\frac{S}{2^4\rho^3 v^4 g^3}\right)^{1/3} \quad Nd=\frac{(1-\beta)\xi^2}{\beta Pr^2} \quad (11)$$

where T_s is the saturated temperature, h_{fg} is the latent heat of phase change, β is the ratio of vapor density to liquid density, p_g is the vapor pressure, and S is the surface tension, u_o is the reference velocity, α is the wave number. Then, by omitting the superscript symbol "*", the above governing equations and the associated boundary conditions become

$$p=\alpha Re^{-1}(r^{-3}\varphi_z)-\alpha^2(r^{-1}\varphi_{zt}+r^{-2}\varphi_z\varphi_{zr} -r^{-3}\varphi_z^2-r^{-2}\varphi_r\varphi_{zz})+\alpha^3 Re^{-1}r^{-1}\varphi_{zzz} \quad (12)$$

$$r^{-1}(r(r^{-1}\varphi_r)_r)_r=\alpha Re(-p_z+r^{-1}\varphi_{rt}+r^{-2}\varphi_z\varphi_{rr} -r^{-3}\varphi_z\varphi_r-r^{-2}\varphi_r\varphi_{rz})-\alpha^2 r^{-1}\varphi_{rzz}+4\Gamma \quad (13)$$

$$r^{-1}(r\theta_r)_r=\alpha Pe(\theta_t-r^{-1}\varphi_r\theta_z+r^{-1}\varphi_z\theta_r)-\alpha^2\theta_{zz} \quad (14)$$

where

$$\Gamma=\left[2(1+a)^2\ln\left(\frac{1+a}{a}\right)-(1+2a)\right]^{-1}. \quad (15)$$

At the cylinder surface ($r=a$)

$$\varphi_r=\varphi_z=\theta=0. \quad (16)$$

At the liquid–vapor interface ($r=a+h$)

$$p+2Re[-\alpha r^{-1}\varphi_{rz}+\alpha r^{-2}\varphi_z+\alpha h_z(-r^{-1}\varphi_{rr} +r^{-2}\varphi_r+\alpha^2 r^{-1}\varphi_{zz})-\alpha^3 r^{-1}\varphi_{rz}h_z^2](1+\alpha^2 h_z^2)^{-1} +WRe^{-5/3}(2\Gamma)^{1/3}[\alpha^2 h_{zz}(1+\alpha^2 h_z)^{-3/2} -r^{-1}(1+\alpha^2 h_z^2)^{-1/2}] +NdRe^{-2}(\theta_r-\alpha^2 h_z\theta_z)^2(1+\alpha^2 h_z^2)^{-1}=0 \quad (17)$$

$$r^{-1}(\varphi_r)_r=\alpha^2 r^{-1}\varphi_{zz}+2\alpha^3(r^{-1}\varphi_z -2\alpha r^{-1}\varphi_{rz})(\alpha^2 h_z^2-1)h_z \quad (18)$$

$$\xi(\theta_r-\alpha^2 h_z\theta_z)+\alpha Pe(r^{-1}\varphi_z+r^{-1}\varphi_r h_z-h_t)=0 \quad (19)$$

$$\theta=1. \quad (20)$$

Since the long wave length modes (i.e. small wave number α) are the most unstable ones, we expand the dimensionless stream function φ, pressure p and temperature θ, in the following form:

$$\varphi = \varphi_0 + \alpha\varphi_1 + O(\alpha^2),$$

$$p = p_0 + \alpha p_1 + O(\alpha^2),$$

$$\theta = \theta_0 + \alpha\theta_1 + O(\alpha^2).$$

Substituting the above expressions into equations (12)–(20). The resulting equations are then solved order by order. In practice, the parameter W is usually of large value, so $\alpha^2 W$ is taken to be of order one. In the liquid film with phase change, Nd is usually small. The zeroth- and first-order solutions (given in Appendix A) are substituted into the dimensionless free surface kinematic equation (19), and yields the nonlinear generalized kinematic equation as

$$h_t + X(h) + A(h)h_x + B(h)h_{xx} + C(h)h_{xxxx} + D(h)h_x^2 + E(h)h_x h_{xxx} = 0. \quad (21)$$

The coefficients of the above equation are given in Appendix B.

STABILITY ANALYSIS

From a simple analysis based on Nusselt's assumption, the variation of film thickness of the base flow is found to be very small for $|\alpha h_x| \ll 1$, so it is reasonable to assume the local dimensionless thickness equals to one. Express the dimensionless film thickness for the perturbed state as:

$$h = 1 + \eta \quad (22)$$

where η is the perturbation of the stationary thickness. The above equation is then substituted into equation (21) and in kept terms up to order of η^3, it yields the evolution of η as follows:

$$\begin{aligned}\eta_t + X'\eta + A\eta_x + B\eta_{xx} + C\eta_{xxxx} + D\eta_x^2 + E\eta_x\eta_{xxx} \\ = -\Bigg[\frac{X''}{2}\eta^2 + \frac{X''}{6}\eta^3 + \left(A'\eta + \frac{A''}{2}\eta^2\right)\eta_x \\ + \left(B'\eta + \frac{B''}{2}\eta^2\right)\eta_{xx} + \left(C'\eta + \frac{C''}{2}\eta^2\right)\eta_{xxxx} \\ + (D + D'\eta)\eta_x^2 + (E + E'\eta)\eta_x\eta_{xx}\Bigg] + O(\eta^4) \quad (23)\end{aligned}$$

where the value of X, A, B, C, D, E and their derivatives are evaluated at $h = 1$.

Linear stability analysis

To study the linear stability analysis, the nonlinear terms of equation (23) are neglected and a linearized equation is obtained as

$$\frac{\partial\eta}{\partial t} + X'\eta + A\frac{\partial\eta}{\partial x} + B\frac{\partial^2\eta}{\partial x^2} + C\frac{\partial^4\eta}{\partial x^4} = 0. \quad (24)$$

Employing the normal mode analysis we assume that

$$\eta = a_1 \exp\,[i(x - dt)] + cc \quad (25)$$

In the above expression, a_1 is the perturbation amplitude and cc is the complex conjugate counterpart. The complex wave celerity d is given by

$$d = d_r + id_i = A + i(B - C - X'). \quad (26)$$

The flow is linearly unstable for $d_i > 0$, and the flow is linearly stable for $d_i < 0$. For $d_i = 0$, it yields the neutral stability curve.

Nonlinear stability analysis

To study the nonlinear stability, the method of multiple scales is used according to

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial t_1} + \varepsilon^2\frac{\partial}{\partial t_2} \quad (27)$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial x_1} \quad (28)$$

$$\eta(\alpha, x, x_1, t, t_1, t_2) = \varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3 \quad (29)$$

where ε is a small parameter, and

$$t_1 = \varepsilon t \quad t_2 = \varepsilon^2 t \quad x_1 = \varepsilon x. \quad (30)$$

Then, equation (23) becomes

$$(L_0 + \varepsilon L_1 + \varepsilon^2 L_2)(\varepsilon\eta_1 + \varepsilon^2\eta_2 + \varepsilon^3\eta_3) = -\varepsilon^2 N_2 - \varepsilon^3 N_3 \quad (31)$$

where

$$L_0 = \frac{\partial}{\partial t} + X' + A\frac{\partial}{\partial x} + B\frac{\partial^2}{\partial x^2} + C\frac{\partial^4}{\partial x^4} \quad (32)$$

$$L_1 = \frac{\partial}{\partial t_1} + A\frac{\partial}{\partial x_1} + 2B\frac{\partial}{\partial x}\frac{\partial}{\partial x_1} + 4C\frac{\partial^3}{\partial x^3}\frac{\partial}{\partial x_1} \quad (33)$$

$$L_2 = \frac{\partial}{\partial t_2} + B\frac{\partial^2}{\partial x_1^2} + 6C\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial x_1^2} \quad (34)$$

$$N_2 = \frac{X''}{2}\eta_1^2 + A'\eta_1\eta_{1x} + B'\eta_1\eta_{1xx} + C'\eta_1\eta_{1xxxx} + D\eta_{1x}^2 + E\eta_{1x}\eta_{1xxx} \quad (35)$$

$$\begin{aligned}N_3 = {} & X''\eta_1\eta_2 + A'(\eta_1\eta_{2x} + \eta_{1x}\eta_2 + \eta_1\eta_{1x_1}) \\ & + B'(\eta_1\eta_{2xx} + 2\eta_1\eta_{1xx_1} + \eta_{1xx}\eta_2) + C'(\eta_1\eta_{2xxxx} \\ & + 4\eta_1\eta_{1xxxx_1} + \eta_{1xxxx}\eta_2) + D(2\eta_{1x}\eta_{2x} \\ & + 2\eta_{1x}\eta_{1x_1}) + E(\eta_{1x}\eta_{2xxx} + 3\eta_{1x}\eta_{1xxx_1} + \eta_{1xxx}\eta_{2x} \\ & + \eta_{1xxx}\eta_{1x_1}) + \tfrac{1}{6}X'''\eta_1^3 + \tfrac{1}{2}A''\eta_1^2\eta_{1x} + \tfrac{1}{2}B\eta_1^2\eta_{1xx} \\ & + \tfrac{1}{2}C''\eta_1^2\eta_{1xxxx} + D'\eta_1\eta_{1x}^2 + E'\eta_1\eta_{1x}\eta_{1xxx}. \quad (36)\end{aligned}$$

Equation (31) is solved order by order. The equation of $O(\varepsilon)$ is $L_0\eta_1 = 0$. The solution is of the following form:

$$\eta_1 = a_1(x_1, t_1, t_2)\exp\,[i(x - d_r t)] + cc. \quad (37)$$

The solution of η_2 and the secular condition for the equation of $O(\varepsilon^2)$ are

$$\eta_2 = ea_2^2 \exp[2i(x-d_r t)]+cc \tag{38}$$

$$\frac{\partial a_2}{\partial t_2}+D_1\frac{\partial a_2}{\partial x_1^2}-\varepsilon^{-2}d_i a_2+(E_1+iF_1)a_2^2\bar{a}_2 = 0 \tag{39}$$

where

$$e = e_r+ie_i = (16C-4B+X)^{-1}\left(\frac{X^{-1}}{2}-B'+C'-D+E-iA'\right)$$
$$D_1 = B-6C$$
$$E_1 = (X''-5B'+17C'+4D-10E)e_r-A'e_i+(\tfrac{1}{2}X'''-\tfrac{3}{2}B''+\tfrac{3}{2}C''+D'-E')$$
$$F_1 = (X''-5B'+17C'+4D-10E)e_i+A'e_r+\tfrac{1}{2}A''. \tag{40}$$

Equation (39) would be used to investigate the weakly nonlinear behavior of fluid film flow.

For a filtered wave, there is no spatial modulation, the solution of equation (38) may be written as

$$a_2 = a_o\exp(-ibt_2). \tag{41}$$

Substituting the above equation into equation (39) and neglecting the second term, one can obtain the following equations:

$$\frac{\partial a_o}{\partial t_2} = (\varepsilon^{-2}d_i-E_1 a_o^2)a_o \tag{42}$$

$$\frac{\partial[b(t_2)t_2]}{\partial t_2} = F_1 a_o^2. \tag{43}$$

Equation (42) is the so-called Ginzburg–Landau equation. Of course, if E_1 were zero, it can be reduced to the equation that is obtained from the linear theory. The second term of the right-hand side of equation (42) is due to the nonlinearity, Nd may moderate or accelerate the exponential growth of the linear disturbance according to the signs of d_i and E_1. Equation (43) is the modification of the wave speed of the infinitesimal disturbance due to the nonlinear effect. In the linearly unstable region ($d_i > 0$), the condition for the existence of a supercritical stable wave is $E_1 > 0$, and the final amplitude (εa_0) is obtained as follows:

$$\varepsilon a_o = \left(\frac{d_i}{E_1}\right)^{1/2} \tag{44}$$

and the nonlinear wave speed is given as

$$Nc_r = \varepsilon^2 b = d_r+d_i\left(\frac{F_1}{E_1}\right)^{1/2}. \tag{45}$$

On the other hand, in the linearly stable region ($d_i < 0$), if $E_1 < 0$, the film flow presents the behavior of subcritical instability, and εa_0 is the threshold amplitude.

RESULTS AND DISCUSSIONS

The linear stability analysis yields the neutral stability curve which is determined by $\alpha d_i = 0$. The α-Re plane is separated into two regions. One is the linearly stable region ($\alpha d_i < 0$) where small disturbances decay with time and the other is the linearly unstable region ($\alpha d_i > 0$) in which small perturbations will grow as time increases. In order to study the effect of radius on the stability of film flow, some of the nondimensional parameters were fixed to be constant values in all numerical calculations, i.e. dimensionless surface tension, $W = 6173.5$; the Jakob number, $\xi = 0.0872$; and Prandtl number, $Pr = 2.62$. The results obtained for the case of plane flow, by setting $a \to \infty$, are generally agreed with those of the previous study by Lin and Weng [17].

Linear stability results

Figures 2(a–c) show the neutral stability curves of condensation film flow for different values of radius, i.e. $a = 10, 16, \infty$, respectively. It is indicated that the region of the linearly stable area ($\alpha d_i < 0$) will be expanded when the value of the radius is increased.

Nonlinear stability results

The nonlinear stability analysis is used to study whether the finite-amplitude disturbance in the linearly stable region will cause instability (subcritical instability) and to study whether the subsequent nonlinear evolution of disturbance in the linearly unstable region will redevelop into a new equilibrium state with a finite amplitude (supercritical stability) or grow to be unstable. By inspecting of the nonlinear amplitude, equation (44), one can find that the negative value of E_1 will make the system become unstable. Such type of instability in the linear stable region is called subcritical instability; i.e. the amplitude of disturbance is larger than the threshold amplitude, and causes the system to reach an explosive state, although the prediction from using the linear theory is stable.

The hatched areas in Figs. 2(a–c) near the neutral stability curve reveal that both subcritical instability ($d_i < 0$, $E_1 < 0$) and the explosive solution ($d_i > 0$, $E_1 < 0$) are all possible for the larger values of a in the film flow.

From the nonlinear stability analysis for subcritical instability, it is shown that if the finite amplitude of the initial disturbance is greater than the threshold amplitude, the system will be unstable. Figure 3 displays the threshold amplitude in the subcritical unstable region with different radius values for the case of $Re = 10$. It is found that the threshold amplitude will become larger for the increasing cylinder radius, then the film flow will be more stable.

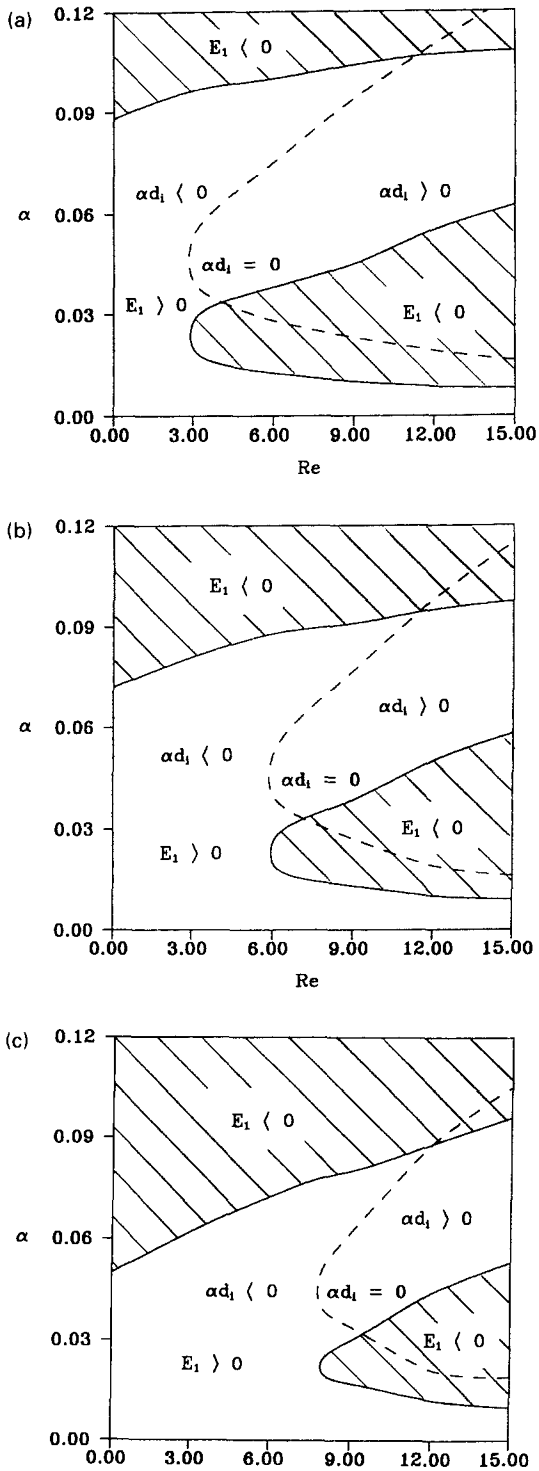

Fig. 2. Stability curve of condensate film flow for (a) $a = 10$, (b) $a = 16$ and (c) $a \to \infty$.

In the linearly unstable region, the linear amplification rate is positive, while the nonlinear amplification rate is negative. Therefore, the linearly infinitesimal disturbance in the unstable region will not grow to infinity, but, rather reaches to an equilibrium

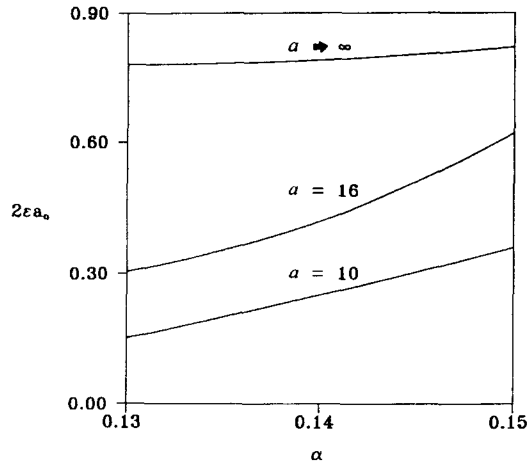

Fig. 3. Threshold amplitude in the subcritical unstable region with different radius values for $Re = 10$.

amplitude that is obtained from equation (44). Figure 4 displays the supercritical stable amplitude with different values of radius for $Re = 5$. It is found that the increase of the radius of cylinder will lower the threshold amplitude; and the flow will be more stable.

The wave speed predicted by the linear theory, given in the equation (26), will not change for all wave numbers; but the nonlinear wave speed, given by equation (45), can be influenced by wave numbers. The variations of nonlinear wave speed with respect to wave number for several values of a are shown in Fig. 5. It is found that both the linear and nonlinear wave speeds are decreased with the increasing radius.

From above discussions, it can be found that a cylinder with a smaller radius makes the flow more unstable. This is due to the surface tension of the lateral curvature. In equation (17), the streamwise surface tension term, $WRe^{-5/3}(2\Gamma)^{1/3}\alpha^2 h_{zz}(1+\alpha^2 h_z)^{-3/2}$, is independent of the value of r, but the lateral surface tension term, $WRe^{-5/3}(2\Gamma)^{1/3}r^{-1}(1+\alpha^2 h_z^2)^{-1/2}$, is inverse to the value of r. When the film flows down

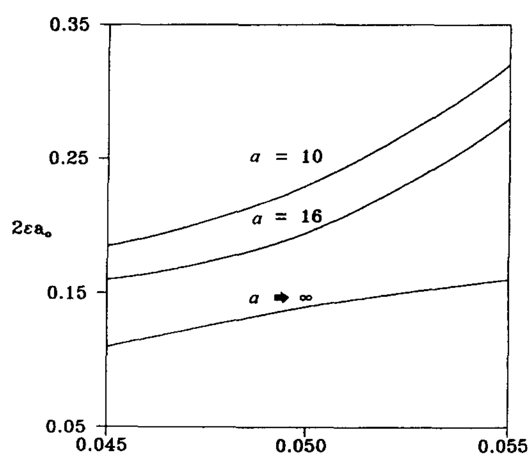

Fig. 4. Threshold amplitude in the supercritical stable region with different radius values for $Re = 5$.

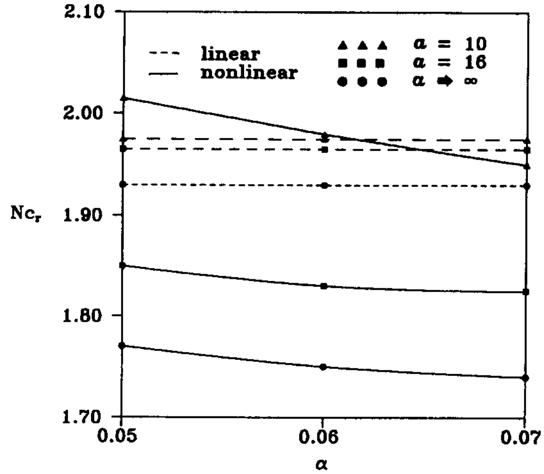

Fig. 5. Speeds of linear and nonlinear wave with different radius values for $Re = 10$.

the outer surface of the cylinder with a smaller radius, the surface tension term of the lateral curvature will become larger. Therefore, it has a destabilizing effect. This destabilizing effect occurs because the radius of the trough of waves have a smaller value than that at the crest of the waves, and the surface tension will produce large capillary pressure at the smaller radius of the curvature. This induces the capillary pressure force tending to move the fluid trough to the crest, thus increasing the amplitude of the wave.

Figure 6 can show that condensate film flow ($\xi > 0$) is more stable than the isothermal film flow ($\xi = 0$); the nonlinear speed for condensate flow is lower than that of isothermal flow, no matter what the value of radius is, because increasing the condensate rate at the wave trough will decrease the wave amplitude, and such effect tends to stabilize the film.

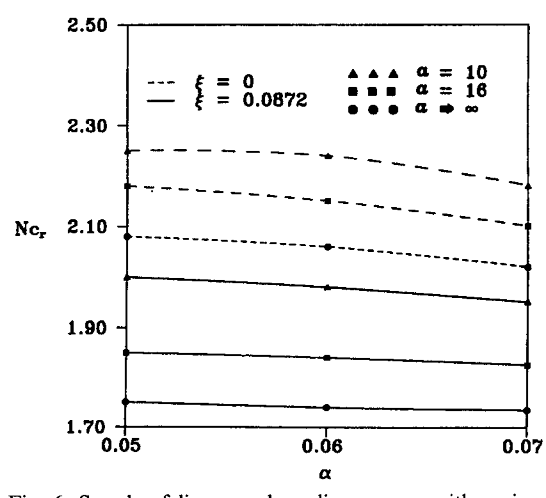

Fig. 6. Speeds of linear and nonlinear wave with various phase change numbers and different radius values for $Re = 10$.

CONCLUSIONS

Weakly nonlinear stability of a condensate film flowing down the outer surface of a vertical cylinder is investigated by the perturbation method. The effect of phase change is taken into account in the interfacial boundary conditions, and a nonlinearly generalized kinematic equation is obtained.

It is shown from linear stability analysis that, the cylindrical curvature has a destabilizing effect because an increase of value of the radius will expand the linearly stable region. Obviously, the linear stability analysis gives a statement of the qualitative tendency of the film flow's dynamic behavior, but not of its finite amplitude. Only from the nonlinear stability analysis of the film flow can the finite amplitudes of stability be obtained.

The method of multiple scales is used for the weak nonlinear stability analysis. It indicates that supercritical stability in the linearly unstable region exists where the infinitesimal disturbance will redevelop into a new equilibrium finite amplitude. Also, there exists a subcritical instability in the linear stable region. The threshold amplitude in the subcritical unstable region will be reinforced with the increase of the cylinder's radius. In the mean time, such an increase will also reduce the amplitude of the supercritical stable wave and nonlinear wave speed.

We conclude that the flow will be more stable with the increase of the cylinder's radius.

REFERENCES

1. W. Nusselt, Die Oberflachenkondensation des Wasserdampfes, *Z. VDI* **50,** 541–546 (1916).
2. S. G. Bankoff, Stability of liquid flow down a heated inclined plane, *Int. J. Heat Mass Transfer* **14,** 377–385 (1971).
3. E. Marschall and C. Y. Lee, Stability of condensate flow down a vertical wall, *Int. J. Heat Mass Transfer* **16,** 41–48 (1973).
4. S. P. Lin, Stability of liquid flow down a heated inclined plane, *Lett. Heat Mass Transfer* **2,** 361–370 (1975).
5. M. Ünsal and W. C. Thomas, Linearized stability analysis of film condensation, *J. Heat Transfer* **100,** 629–634 (1978).
6. J. P. Burelbach, S. G. Bankoff and S. H. Davis, Nonlinear stability of evaporating/condensing liquid films, *J. Fluid Mech.* **195,** 463–494 (1988).
7. S. W. Joo, S. H. Davis and S. G. Bankoff, Long-wave instabilities of heated falling films: two-dimensional theory of uniform layers, *J. Fluid Mech.* **230,** 117–146 (1991).
8. D. J. Benney, Long waves on liquid film, *J. Math. Phys.* **45,** 150–155 (1966).
9. B. E. Anshus, Finite amplitude wave flow of a thin film on a vertical wall, Ph.D. dissertation, University of California, Berkeley (1965).
10. S. P. Lin, Finite amplitude side-band stability of a viscous film, *J. Fluid Mech.* **63,** 417–419 (1974).
11. M. Ünsal and W. C. Thomas, Nonlinear stability of film condensation, *J. Heat Transfer* **102,** 483–488 (1980).
12. C. C. Hwang and C. I. Weng, Finite-amplitude stability analysis of liquid films down a vertical wall with and without interfacial phase change, *Int. J. Multiphase Flow* **13,** 803–814 (1987).
13. J. S. Tsai, C. I. Hung and C. K. Chen, Nonlinear hydro-

magnetic stability analysis of condensation film flow down a vertical plane, *Acta Mech.* (in press).
14. H. C. Chang, Wave evolution on a falling film, *Ann. Rev. Fluid Mech.* **26**, 103–136 (1994).
15. S. P. Lin and W. C. Liu, Instability of film coating of wires and tubes, *AIChE J.* **21**, 775–782 (1975).
16. W. B. Krantz and R. L. Zollars, The linear hydrodynamic stability of film flow down a vertical cylinder, *AIChE J.* **22**, 930–934 (1976).
17. J. S. Lin and C. I. Weng, Linear stability analysis of condensate film flows down a vertical cylinder, *Chem. Engng Commun.* **57**, 263–276 (1987).
18. P. Rosenau and A. Oron, Evolution and breaking of liquid film flowing on a vertical cylinder, *Phys. Fluids* **A1**, 1763–1766 (1989).
19. L. A. Davalos-Orozco and G. Ruiz-Chavarria, Hydrodynamic instability of a fluid layer flowing down a rotating cylinder, *Phys. Fluids* **A5**, 2390–2404 (1993).

APPENDIX A

Zeroth-order solution

$$\varphi_0 = \Gamma\left[\tfrac{1}{4}(r^4+a^4)-\tfrac{1}{2}a^2r^2+\tfrac{1}{2}q^2(r^2-a^2)-q^2r^2\ln\left(\frac{r}{a}\right)\right]$$

$$p_0 = 2\alpha^2 WRe^{-5/3}(2\Gamma)^{1/3}h_{zz}+2WRe^{-5/3}(2\Gamma)^{1/3}$$

$$\theta_0 = \left(\ln\frac{r}{a}\right)(\ln Q)^{-1}$$

where

$$q = h+a$$

$$Q = \frac{a+h}{a}.$$

First-order solution

$$\varphi_1 = A_1r^6+A_2r^4+A_3r^2+A_4$$

$$p_1 = -2Re^{-1}h_z\Gamma\left[2q\ln\left(\frac{r}{a}\right)+q-a^2q^{-1}\right]$$

$$\theta_1 = B_1r^4+B_2r^2+B_3\ln r+B_4$$

where

$$A_1 = Re\Gamma^2qh_z\left[\frac{13}{288}-\frac{1}{48}\ln\left(\frac{r}{a}\right)\right]$$

$$\begin{aligned}A_2 = {} & \tfrac{1}{16}2WRe^{-2/3}(2\Gamma)^{1/3}(\alpha^2h_{zzz}+q^{-2}h_z) \\ & +Reqh_zh_{0t}\left[\frac{5}{16}-\frac{1}{4}\ln\left(\frac{r}{a}\right)\right]+Re\Gamma^2\left\{q^3h_z\left[-\frac{1}{4}\left(\ln\left(\frac{r}{a}\right)\right)^2\right.\right. \\ & \left.\left.+\frac{3}{4}\ln\left(\frac{r}{a}\right)-\frac{13}{16}\right]+qa^2h_z\left[-\frac{3}{32}-\frac{1}{8}\ln\left(\frac{r}{a}\right)\right]\right\} \\ & -\tfrac{1}{8}(\beta-1)NdRe^{-1}q^{-1}(\ln Q)^{-1}[q^{-2}(\ln Q)^{-1} \\ & +q^{-2}(\ln Q)^{-2}]h_z\end{aligned}$$

$$\begin{aligned}A_3 = {} & \tfrac{1}{8}2WRe^{-2/3}(2\Gamma)^{1/3}(\alpha^2h_{zzz}+q^{-2}h_z)\left[q^2-a^2\right. \\ & \left.-2q^2\ln\left(\frac{r}{a}\right)\right]-\tfrac{1}{4}(\beta-1)NdRe^{-1}q^{-1}(\ln Q)^{-1}[q^{-2}(\ln Q)^{-1} \\ & +q^{-2}(\ln Q)^{-1}]\left[q^2-a^2-2q^2\ln\left(\frac{r}{a}\right)\right] \\ & +\Gamma Reh_zq\left[q^2(\tfrac{1}{2}\ln Q-\tfrac{1}{4})\left(2\ln\left(\frac{r}{a}\right)-1\right)-\tfrac{1}{2}a^2\right] \\ & +\Gamma^2Reh_z\left\{qa^4\left[\tfrac{1}{4}-\tfrac{1}{4}\ln^2a+\tfrac{1}{4}\ln q-\tfrac{1}{2}\ln q\ln\left(\frac{r}{a}\right)\right.\right. \\ & \left.+\tfrac{1}{4}(\ln r)^2-\tfrac{1}{4}\ln r\right]+q^3a^2\left[1-\tfrac{1}{2}(\ln a)^2+\tfrac{1}{3}(\ln a)^3\right. \\ & -\tfrac{1}{4}\ln Q+\tfrac{1}{2}\ln q+\tfrac{1}{4}\ln^2q-\tfrac{1}{2}\ln q\ln a \\ & +\tfrac{1}{2}\ln Q\ln\left(\frac{r}{a}\right)-\ln q\ln\left(\frac{r}{a}\right)-\tfrac{1}{2}\ln^2q\ln\left(\frac{r}{a}\right) \\ & +\ln q\ln a\ln\left(\frac{r}{a}\right)+\frac{3}{4}\left(\ln\frac{r}{a}\right)+\tfrac{1}{4}(\ln r)^2-\tfrac{1}{4}\ln r+\tfrac{1}{6}(\ln r)^3 \\ & \left.-\tfrac{1}{2}\ln a(\ln r)^2+\tfrac{1}{2}\ln a\ln r-\tfrac{1}{4}\ln a\right] \\ & +q^5\left[\tfrac{5}{8}\ln Q-\tfrac{1}{2}(\ln Q)^2-\tfrac{13}{32}-\tfrac{5}{4}\ln Q\ln\left(\frac{r}{a}\right)\right. \\ & \left.\left.+(\ln Q)^2\left(\ln\frac{r}{a}\right)+\tfrac{13}{16}\left(\ln\frac{r}{a}\right)\right]\right\}\end{aligned}$$

$$\begin{aligned}A_4 = {} & \tfrac{1}{16}2WRe^{-2/3}(2\Gamma)^{1/3}[\alpha^2h_{zzz}a^2(a^2-2q^2) \\ & +a^2h_z(q^{-2}a^2-2)]+Re\Gamma^2h_z\{q^5a^2[\tfrac{13}{32}-\tfrac{5}{8}\ln Q \\ & +\tfrac{1}{2}\ln^2Q]+q^3a^4[-\tfrac{3}{16}-\tfrac{1}{4}\ln Q-\tfrac{1}{4}\ln^2Q] \\ & +qa^6[-\tfrac{29}{144}-\tfrac{1}{4}\ln Q]\},\end{aligned}$$

$$B_1 = \tfrac{1}{16}Pe\Gamma h_zq^{-1}(\ln Q)^{-2}\left[\ln\frac{r}{a}-\frac{1}{2}\right]$$

$$\begin{aligned}B_2 = {} & \tfrac{1}{4}Peh_zq^{-1}(\ln Q)^{-2}\left[1-\ln\left(\frac{r}{a}\right)\right]+Pe\Gamma h_zq^{-1}(\ln Q)^{-2} \\ & \times\left\{\frac{1}{4}a^2\left[1-\ln\left(\frac{r}{a}\right)\right]-q^2\left[\frac{1}{2}\left(\ln\frac{r}{a}\right)^2-\left(\ln\frac{r}{a}\right)+\frac{3}{4}\right]\right\} \\ & +Pe\Gamma h_zq\ln^{-1}Q\left[\frac{3}{4}-\frac{1}{2}\left(\ln\frac{r}{a}\right)\right]\end{aligned}$$

$$\begin{aligned}B_3 = {} & \tfrac{1}{4}Peh_{0t}q^{-1}(\ln Q)^{-3}(a^2+q^2\ln Q-q^2) \\ & +Pe\Gamma h_z\{q^{-1}(\ln Q)^{-3}[a^2(\tfrac{7}{32}a^2-q^2+\tfrac{1}{4}q^2\ln Q) \\ & +q^4(\tfrac{25}{32}-\tfrac{17}{16}\ln Q)+\tfrac{1}{2}(\ln Q)^2] \\ & +q(\ln Q)^{-2}[a^2(\tfrac{3}{4}-\tfrac{1}{2}(\ln a)^2+\tfrac{1}{2}(\ln q)^2 \\ & -\tfrac{1}{2}\ln r\ln Q)+q^2(\tfrac{1}{2}\ln Q-\tfrac{3}{4})]\}\end{aligned}$$

$$\begin{aligned}B_4 = {} & -[B_3+\tfrac{1}{2}Pe\Gamma h_zq(\ln Q)^{-1}a^2\ln r]\ln a \\ & -\tfrac{1}{4}Peh_{0t}q^{-1}(\ln Q)^{-2}a^2 \\ & -Pe\Gamma h_z\{q^{-1}(\ln Q)^{-2}(\tfrac{7}{32}a^4-\tfrac{3}{4}q^2a^4) \\ & -q(\ln Q)^{-1}[\tfrac{3}{4}a^2-\tfrac{1}{2}a^2(\ln a)^2]\}.\end{aligned}$$

where

$$h_{0t} = 2\Gamma(q^2-a^2-2q^2\ln Q)h_z+\frac{\xi}{\alpha Pe}(1+\alpha^2h_z^2)q^{-1}(\ln Q)^{-1}$$

APPENDIX B

$$X(h) = \frac{\xi}{\alpha Pe}[K_1(h) + \xi K_2(h)] \tag{B1}$$

where

$$K_1(h) = -q^{-1}(\ln Q)^{-1}$$

$$K_2(h) = -[\tfrac{1}{4}q^{-3}(\ln Q)^{-4}(a^2 - q^2 + q^2 \ln Q) - q^{-1}(\ln Q)^{-3}(\tfrac{1}{2}\ln Q - \tfrac{1}{4})]$$

$$A(h) = K_3(h) - \xi K_4(h) + \frac{\xi}{Pr}K_5(h) \tag{B2}$$

where

$$K_3(h) = 2\Gamma(a^2 - q^2 + 2q^2 \ln Q)$$

$$K_4(h) = \Gamma\{(\tfrac{1}{4}q^2 + \tfrac{1}{2}a^2 - a^2 \ln q)(\ln Q)^{-1} + [-\tfrac{3}{4}a^2 - \tfrac{3}{8}q^2 + \tfrac{1}{2}a^2(\ln^2 q - \ln^2 a)](\ln Q)^{-2} + \tfrac{9}{32}(q^2 - a^4 q^{-2})(\ln Q)^{-3}\}$$

$$K_5(h) = \Gamma\{4q^2 - a^2 + (\tfrac{3}{2}a^2 - \tfrac{9}{4}q^2)(\ln Q)^{-1} + (\tfrac{9}{16}q^2 + \tfrac{3}{16}a^4 q^{-2} - \tfrac{3}{4}a^2)(\ln Q)^{-2} - 4q^2 \ln Q\}.$$

$$B(h) = 2\alpha W Re^{-2/3}(2\Gamma)^{1/3}K_6(h) + \alpha Re K_7(h) + \alpha(\beta - 1)Nd Re^{-1}K_8(h) \tag{B3}$$

where

$$K_6(h) = \tfrac{1}{16}(4a^2 q^{-1} + 4q \ln Q - 3q - a^4 q^{-3})$$

$$K_7(h) = \Gamma^2\{q^6[-3\ln^3 Q + 5\ln^2 Q - \tfrac{31}{12}\ln Q - \tfrac{7}{144}] + q^4 a^2[-\tfrac{1}{3}\ln^3 Q - \tfrac{7}{2}\ln^2 Q + \tfrac{13}{2}\ln Q - \tfrac{21}{16}] + q^2 a^4[-\tfrac{1}{2}\ln^2 Q - 2\ln Q + \tfrac{31}{16}] + a^6[-\tfrac{1}{4}\ln Q - \tfrac{83}{144}]\}$$

$$K_8(h) = \tfrac{1}{8}q^{-1}\ln^{-1} Q[q^{-2}\ln^{-1} Q + q^{-2}\ln^{-2} Q](3q^3 a^{-1} - 4q^3 \ln Q - 4a^2 q).$$

$$C(h) = 2\alpha^2 W Re^{-2/3}(2\Gamma)^{1/3}K_9(h) \tag{B4}$$

where

$$K_9(h) = q^3(\tfrac{1}{4}\ln Q - \tfrac{3}{16}) + \tfrac{1}{4}a^2 q - \tfrac{1}{16}a^4 q^{-1}.$$

$$D(h) = 2\alpha W Re^{-2/3}(2\Gamma)^{1/3}K_{10}(h) + \alpha Re K_{11}(h) + \frac{\alpha\xi}{Pe}[K_{12}(h) + \xi K_{13}(h)] \tag{B5}$$

where

$$K_{10}(h) = \tfrac{1}{16}(-4a^2 q^{-2} + 4\ln Q + 1 + 3a^4 q^{-4})$$

$$K_{11}(h) = \Gamma^2\{q^5[-18(\ln^3 Q) + 21(\ln^2 Q) - \tfrac{11}{2}\ln Q - \tfrac{23}{8}] + q^3 a^2[-\tfrac{4}{3}\ln^3 Q - 15\ln^2 Q + 19\ln Q + \tfrac{5}{4}] + q a^4[-\ln^2 Q - 5\ln Q + \tfrac{15}{8}] + q^{-1}a^6[-\tfrac{1}{4}]$$

$$K_{12}(h) = -q^{-1}(\ln Q)^{-1}$$

$$K_{13}(h) = q^{-1}(\ln Q)^{-3}(\tfrac{1}{2}\ln Q - \tfrac{1}{4}) - q^{-3}(\ln Q)^{-4}(\tfrac{1}{4}a^2 + \tfrac{1}{4}q^2 \ln Q - \tfrac{1}{4}q^2).$$

$$E(h) = 2\alpha^2 W Re^{-2/3}(2\Gamma)^{1/3}K_{14}(h) \tag{B6}$$

where

$$K_{14}(h) = q^2 \ln Q - \tfrac{1}{2}q^2 + \tfrac{1}{2}a^2.$$